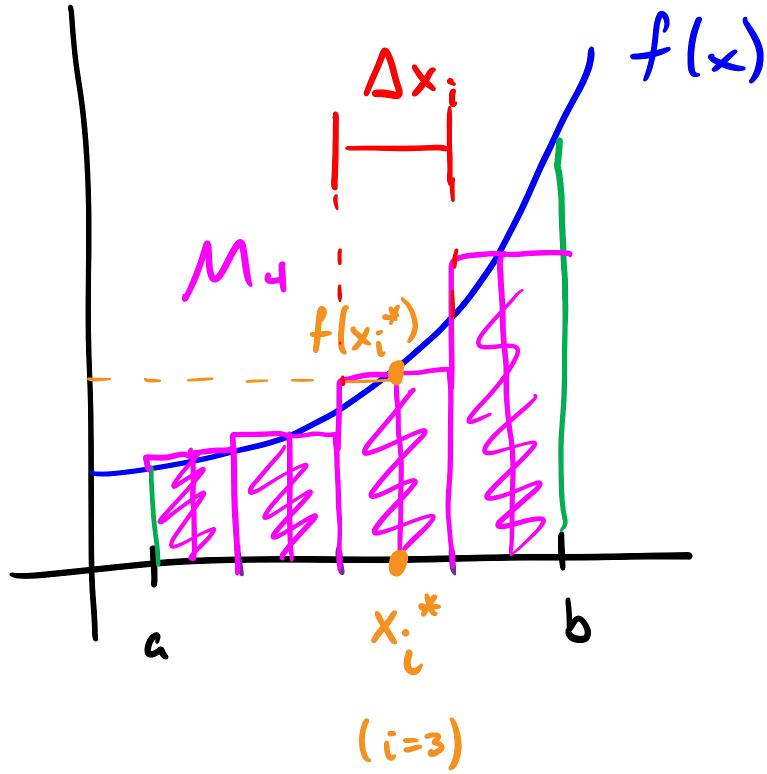


Area of i-th rectangle is $f(x_i^*)\cdot\Delta x_i$

Total area with n rectangles : $f(x_1^*)\Delta x_1 + f(x_2^*)\Delta x_2 + \cdots + f(x_n^*)\Delta x_n$

$$= \underbrace{\sum_{i=1}^{n} f(x_i^*)\,\Delta x_i^*}$$: Riemann Sum

Ex of Σ-notation

$$\sum_{i=1}^{5}(2i-3) = (2\cdot1-3)+(2\cdot2-3)+(2\cdot3-3)+(2\cdot4-3)+(2\cdot5-3)$$

$$= -1+1+3+5+7 = 15$$

$$\sum_{i=1}^{n} f(x_i^*)\,\Delta x_i$$

Appoximate the area under $f(x)$ on $[a,b]$

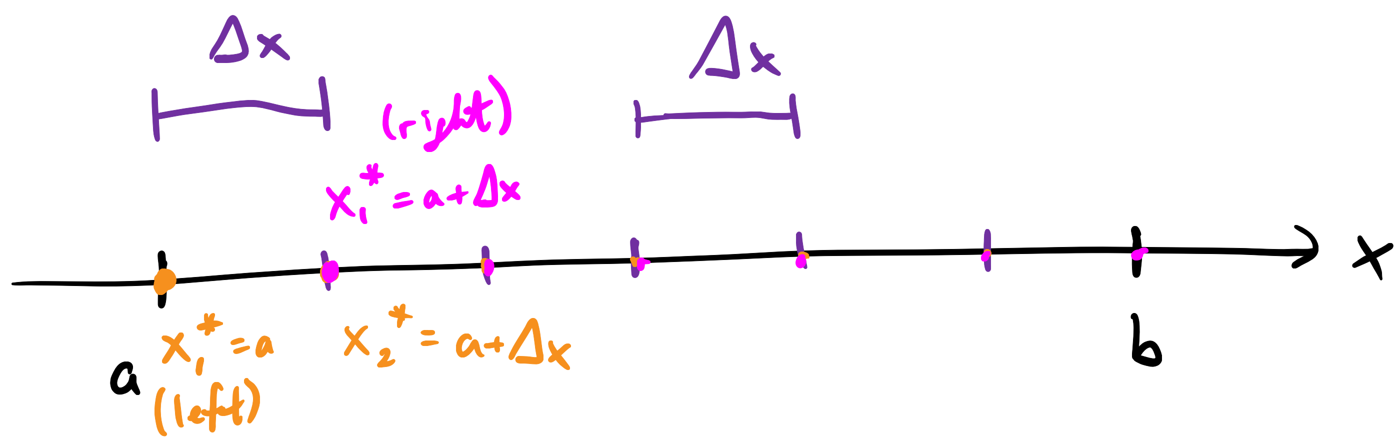
- Cut $[a,b]$ in n equal pieces:

  width of each segment: $\Delta x = \dfrac{b-a}{n}$

- left endpoints: $x_i^* = a + (i-1)\Delta x$

  right endpoints: $x_i^* = a + i\Delta x$

  midpoints: $x_i^* = a + (i-\frac{1}{2})\Delta x$



Ex: Approximate the area under $f(x)=x^3$ with n rectangles on $[1,3]$.

Sol: $\Delta x = \frac{3-1}{n} = \frac{2}{n}$

$$x_i^* = a + i\Delta x = 1 + i\cdot\frac{2}{n} = 1 + \frac{2i}{n}$$

$$f(x_i^*) = \left(1+\frac{2i}{n}\right)^3$$

$$= 1^3 + 3(1)^2\cdot\frac{2i}{n} + 3\cdot 1\cdot\left(\frac{2i}{n}\right)^2 + \left(\frac{2i}{n}\right)^3$$

$$= 1 + \frac{6i}{n} + \frac{12i^2}{n^2} + \frac{8i^3}{n^3}$$

$$A \approx \sum_{i=1}^{n} f(x_i^*)\Delta x$$

$$= \sum_{i=1}^{n}\left(1 + \frac{6i}{n} + \frac{12i^2}{n^2} + \frac{8i^3}{n^3}\right)\frac{2}{n}$$

$$= \sum_{i=1}^{n}\left(\frac{2}{n} + \frac{12}{n^2}i + \frac{24}{n^3}i^2 + \frac{16}{n^4}i^3\right)$$

$$= \frac{2}{n}\boxed{\sum_{i=1}^{n} 1} + \frac{12}{n^2}\boxed{\sum_{i=1}^{n} i} + \frac{24}{n^3}\boxed{\sum_{i=1}^{n} i^2} + \frac{16}{n^4}\boxed{\sum_{i=1}^{n} i^3}$$

$$= \frac{2}{n}\cdot n + \frac{12}{n^2}\,\frac{n(n+1)}{2} + \frac{24}{n^3}\,\frac{n(n+1)(2n+1)}{6} + \frac{16}{n^4}\cdot\left(\frac{n(n+1)}{2}\right)^2$$

$$= 2 + \frac{6n(n+1)}{n^2} + \frac{4n(n+1)(2n+1)}{n^3} + \frac{4n^2(n+1)^2}{n^4}$$

Q: What is the exact area?

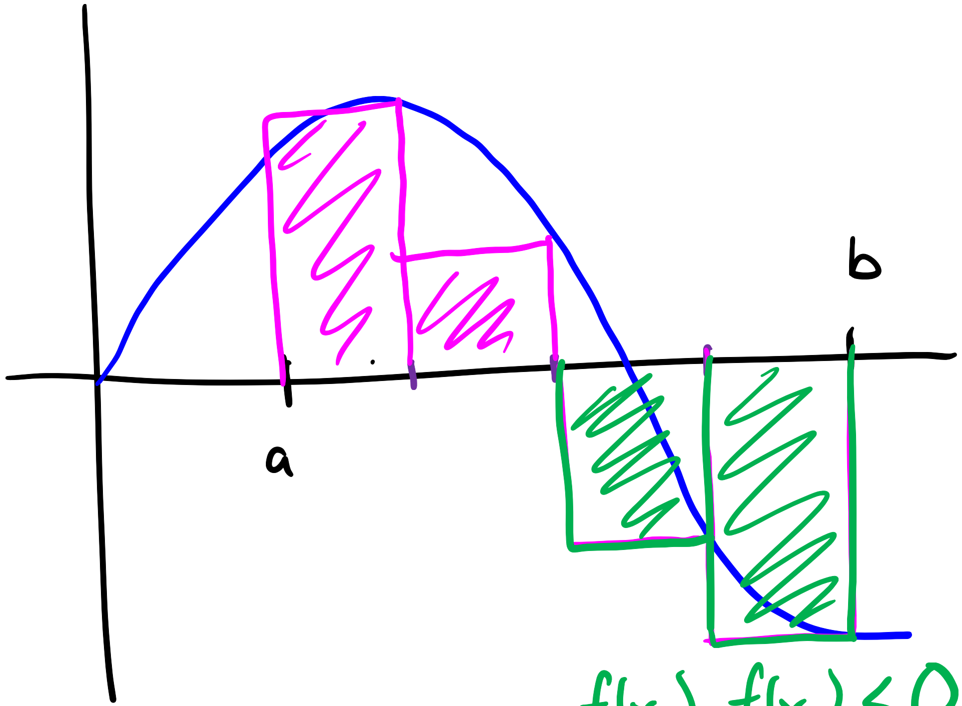
$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x_i = \int_a^b f(x)\,dx$$

Definite integral of $f(x)$ from a to b

So,

$$\int_1^3 x^3\,dx = \lim_{n\to\infty}\left(2 + \underbrace{\frac{6n(n+1)}{n^2}}_{6n^2} + \underbrace{\frac{4n(n+1)(2n+1)}{n^3}}_{8n^3} + \underbrace{\frac{4n^2(n+1)^2}{n^4}}_{4n^4}\right)$$

$$= 2 + 6 + 8 + 4 = \boxed{20}$$



$f(x_3), f(x_4) < 0 \rightsquigarrow$ $f(x_3)\Delta x < 0$

$f(x_4)\Delta x < 0$

$$\sum_{i=1}^{4} f(x_i^*)\Delta x_i$$

$\int_a^b f(x)\,dx =$ "net area"

above x-axis – "positive area"

below x-axis – "negative area"